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The Transition to Unigroups

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A *unigroup* is defined to be a partially ordered abelian group with a distinguished generative universal order unit. Virtually any structure that has been proposed for the logic, sharp or unsharp, of a physical system can be represented by the order interval in a unigroup. Furthermore, probability states correspond to positive, normalized, real-valued group homomorphisms, and physical symmetries correspond to unigroup automorphisms. We show that the category of unigroups admits arbitrary products and coproducts. A new class of interval effect algebras called *Heyting effect algebras* (HEAs) is introduced and studied. Among other things, an HEA is both a Heyting algebra and a BZ-lattice in which the sharp elements are precisely the central elements. Certain HEAs arise naturally from partially ordered abelian groups affiliated with Stone spaces. Using Stone unigroups, we obtain perspicuous representations for certain multivalued logics, including the three-valued logic of conditional events utilized by Goodman, Nguyen, and Walker in their study of logic for expert systems.

1. INTRODUCTION

At the September 1992 meeting of the International Quantum Structures Association (IQSA) in Castiglioncello, one of us (R.J.G.) presented a paper entitled "The transition to orthoalgebras." Orthoalgebras are now widely regarded as fundamental structures, not only in quantum logic, but in the rapidly developing field of noncommutative measure theory (D'Andrea and DeLucia, 1991a,b; DeLucia and Dvurečenskij, 1993; Rüttimann, 1980, 1989). In the same spirit, two of us (Greechie and Foulis, 1995) advocated a "Transition to effect algebras" in the *Proceedings* of the August 1994 meeting of the IQSA in Prague (Pták, 1995). Effect algebras (Foulis and Bennett, 1994; Foulis *et al.*, 1996; Greechie *et al.*, 1995)—also called D-posets (Dvurečenskij

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and Pulmannová, 1994; Kôpka and Chovanec, 1994) or generalized orthoalgebras (Giuntini and Greuling, 1989)—are playing an ever-increasing role in the foundations of quantum mechanics.

As a follow-up to our talks at the July/August 1996 meeting of the IQSA in Berlin, we now commend partially ordered abelian groups with distinguished generative universal elements—*unigroups* for short—to the attention of the quantum logic community. Whereas the earlier transitions advocated more and more general settings in which to cast the theory, the proposed transition to unigroups actually provides a representation theory for effect algebras associated with physical systems that is considerably stronger than anything available for general effect algebras.

There is no longer any novelty in the observation that, associated with a physical system $\mathcal{S}$ under experimental study there is a so-called *logic* L, the elements of which can be regarded as experimentally testable propositions about $\mathcal{S}$. An *observable* for $\mathcal{S}$ is an L-valued measure $\mathcal{A}$: $\mathcal{B} \to L$ defined on a boolean algebra $\mathcal{B}$, and a *state* for $\mathcal{S}$ induces a probability measure ω: $L \to [0, 1] \subseteq \mathbb{R}$ (Gudder, 1988). All three of the mathematical structures $\mathcal{B}$, L, and $[0, 1]$ are effect algebras, and the mappings $\mathcal{A}$: $\mathcal{B} \to L$, ω: $L \to [0, 1]$, and $\omega \circ A$: $\mathcal{B} \to [0, 1]$ are effect-algebra morphisms. The boolean algebra $\mathcal{B}$ and the scale algebra $[0,1]$ are isomorphic to unit intervals in corresponding unigroups. If there are sufficiently many probability measures ω: $L \to [0, 1]$ to determine the order structure of L, then L, too, is isomorphic to the unit interval in a unigroup (Theorems 2.11, 3.3, and 4.5 below). Morphisms $\mathcal{A}$: $\mathcal{B} \to L$ and ω: $L \to [0, 1]$ then can be extended uniquely to normalized positive group homomorphisms $\mathcal{A}^*$ and ω^* on the corresponding unigroups. (This is the *universal property* from which unigroups derive their name.)

For a quantum mechanical system $\mathcal{S}$ represented in the usual way by a Hilbert space $\mathcal{H}$, the boolean algebra $\mathcal{B}$ is the σ-algebra of real Borel sets, the logic L is the algebra of effect operators on $\mathcal{H}$, the states are represented by density operators on $\mathcal{H}$, and the probability measure ω corresponding to a density W is defined by $\omega(A) = \mathrm{tr}(WA)$ for all $A \in L$. The unigroup for $\mathcal{B}$ is the group $C(X, \mathbb{Z})$ of continuous integer-valued functions on the Stone space X of $\mathcal{B}$; the unigroup for L is the additive group $\mathbb{G}(\mathcal{H})$ of bounded self-adjoint operators on $\mathcal{H}$; and the unigroup for $[0, 1]$ is the additive group $\mathbb{R}$ of real numbers. Effect-algebra automorphisms of L extend uniquely to unigroup automorphisms of $\mathbb{G}(\mathcal{H})$, which in turn correspond uniquely (via Wigner's theorem on symmetry transformations) to unitary or antiunitary operators on the Hilbert space $\mathcal{H}$.

Thus, the program of studying physical systems in terms of observables, experimental propositions, states, and symmetries can be reformulated in terms of unigroups, unigroup homomorphisms, and unigroup automorphisms. In this way, the highly developed theory of partially ordered abelian groups can be brought to bear on the study of the mathematical foundations of the experimental sciences.

2. ORTHOSTRUCTURES AND EFFECT ALGEBRAS

An orthostructure, as defined below, is perhaps the most general *algebraic* structure that can meaningfully carry probability measures. [A correspondingly general *order-theoretic* structure has been introduced by Gudder (1996).]

Definition 2.1. An *orthostructure* is a system $(L, 0, u, \perp, \oplus)$ consisting of a set L, special elements $0, u \in L$ called the *zero* and the *unit*, a binary relation $\perp$ on L called *orthogonality*, and a partially defined binary operation $\oplus$ on L called *orthosummation* such that: (i) for all $p, q \in L$, $p \oplus q$ is defined iff $p \perp q$; (ii) if $p \in L$, then there exist $q, r \in L$ such that $p \perp q, r \perp p$, and $p \oplus q = r \oplus p = u$; and (iii) if $p \in L$, then $p \perp 0$, $0 \perp p$, and $p \oplus 0 = 0 \oplus p = p$.

Example 2.2. Let G be a multiplicatively written group, let $1 \in L \subseteq G$, and let $u \in L$ be such that $p \in L \Rightarrow p^{-1}u, up^{-1} \in L$. For $p, q \in L$, define $p \perp q$ iff $pq \in L$ and, if $pq \in L$, define $p \oplus q := pq$. (The notation $:=$ means equals by definition.) Then $(L, 1, u, \perp, \oplus)$ is an orthostructure.

The following example, which is a special case of Example 2.2, except that the group is additively written, will be of special interest in the sequel.

Example 2.3. Let G be an additively written, partially ordered abelian group and let $u \in G$ with $0 \le u$. Define $L := \{p \in G | 0 \le p \le u\}$. For $p, q \in L$, define $p \perp q$ iff $p + q \in L$ and, if $p + q \in L$, define $p \oplus q := p + q$. Then $(L, 0, u, \perp, \oplus)$ is an orthostructure.

If no confusion threatens, we say that L is an orthostructure when we really mean that $(L, 0, u, \perp, \oplus)$ is an orthostructure.

Definition 2.4. A *sub-orthostructure* of an orthostructure L with unit u is a subset $S \subseteq L$ such that: (i) $0, u \in S$; (ii) if $x, y \in S$ and $x \perp y$, then $x \oplus y \in S$; and (iii) if $x \in S$, there exist $y, z \in S$ such that $x \perp y$, $z \perp x$, and $x \oplus y = z \oplus x = u$.

Evidently, a sub-orthostructure S of an orthostructure L is an orthostructure in its own right, with the same zero and unit as L, with the restriction to S of $\perp$ as orthogonality, and with the restriction of $\oplus$ to S as the orthosummation.

Definition 2.5. If K is an additive abelian group, then a *K-valued measure* on the orthostructure $(L, 0, u, \perp, \oplus)$ is a mapping $\phi: L \to K$ such that, for all $p, q \in L$, $p \perp q \Rightarrow \phi(p \oplus q) = \phi(p) + \phi(q)$. A *probability measure* on $(L, 0, u, \perp, \oplus)$ is an $\mathbb{R}$-valued measure ω such that $0 \leq \omega(p) \leq 1$ for all $p \in L$ and $\omega(u) = 1$.

A *morphism* from an orthostructure $(L, 0, u, \perp, \oplus)$ to an orthostructure $(M, 0, v, \#, \boxplus)$ is understood to be a mapping $\alpha\colon L \to M$ such that (i) $\alpha(0) = 0$, (ii) $\alpha(u) = v$, and (iii) if $p, q \in L$, then $p \perp q \Rightarrow \alpha(p) \# \alpha(q)$ and $\alpha(p \oplus q) = \alpha(p) \boxplus \alpha(q)$. Isomorphisms and automorphisms of orthostructures are defined in the obvious ways.

If K is an additive abelian group, # is the universal relation on K, and $\phi : L \to K$ is a K-valued measure, then ϕ is a morphism from the orthostructure $(L, 0, u, \perp, \oplus)$ to the orthostructure $(K, 0, \phi(u), \#, +)$. Let ω be a probability measure on L, and organize the unit interval $[0, 1]$ in the additive partially ordered abelian group $\mathbb{R}$ into an orthostructure as in Example 2.3. Then ω is a morphism from L to $[0, 1]$.

Definition 2.6. Let Λ be a set of probability measures on the orthostructure $(L, 0, u, \perp, \oplus)$. Say that Λ is a *full* set of probability measures iff, for $p, q \in L$, the condition $\omega(p) + \omega(q) \leq 1$ for all $\omega \in \Lambda$ implies that $p \perp q$. Say that Λ is a *separating* set of probability measures iff, for $p, q \in L$, the condition $\omega(p) = \omega(q)$ for all $\omega \in \Lambda$ implies that $p = q$.

Definition 2.7. Let $(L, 0, u, \perp, \oplus)$ be an orthostructure.

(i) L is *commutative* iff, for $p, q \in L$, $p \perp q \Rightarrow q \perp p$ and $p \oplus q = q \oplus p$.

(ii) L is *associative* iff, for $p, q, r \in L$, $q \perp r$ and $p \perp (q \oplus r) \Rightarrow p \perp q$, $(p \oplus q) \perp r$, and $p \oplus (q \oplus r) = (p \oplus q) \oplus r$.

Definition 2.8. An *effect algebra* (Foulis and Bennett, 1994) is a commutative and associative orthostructure $(L, 0, u, \perp, \oplus)$ satisfying the following conditions:

(i) (*Orthosupplementation law*) For each $p \in L$ there is a unique $q \in L$ such that $p \perp q$ and $p \oplus q = u$.

(ii) (*Zero-unit law*) If $p \in L$ and $u \perp p$, then $p = 0$.

An effect algebra in which the zero and unit coincide, and which therefore consists only of the single element 0, is said to be *degenerate.*

Definition 2.9. Let $(L, 0, u, \perp, \oplus)$ be an effect algebra.

(i) If $p \in L$, the unique element $q \in L$ such that $p \perp q$ and $p \oplus q = u$, called the *orthosupplement* of p, is denoted by $p' := q$.

(ii) If $p,q \in L$, define $p \leq q$ iff there is an element $r \in L$ such that $p \perp r$ and $p \oplus r = q$.

If L is an effect algebra, then L is partially ordered by $\leq$, and $0 \leq p \leq u$ holds for all $p \in L$. Furthermore, for $p, q \in L$, $p \perp q$ iff $p \leq q'$.

Because a sub-orthostructure S of an effect algebra L is again an effect algebra, we refer to S as a *sub-effect* algebra of L.

Definition 2.10. Let L be an effect algebra.

(i) An element $p \in L$ is *isotropic* iff $p \perp p$.

(ii) An element $p \in L$ is *principal* iff, for $q, r \in L$, $q \perp r$ and $q, r \leq p \Rightarrow q \oplus r \leq p$.

(iii) Elements $p, q \in L$ are *disjoint* iff, for all $r \in L$, $r \leq p, q \Rightarrow r = 0$.

(iv) L is *regular* iff every pair of isotropic elements in L is an orthogonal pair (Cattaneo and Nistico, 1989).

An *orthoalgebra* can be defined as an effect algebra with no nonzero isotropic elements. An *orthomodular poset* is the same thing as an effect algebra in which every element is principal. An effect algebra is said to be *lattice ordered* iff it forms a lattice under $\leq$. An *orthomodular lattice* can be defined as a lattice-ordered orthoalgebra. A *boolean algebra* is the same thing as an orthomodular lattice in which disjoint elements are orthogonal. There are orthoalgebras that are not boolean algebras, but in which disjoint elements are orthogonal (Bennett and Foulis, 1993, Section 7).

The simplest example of an effect algebra L with a nonzero isotropic element is obtained by taking G to be the additive group of integers and $u = 2$ in Example 2.3. The resulting three-element effect algebra, which we call C_2 (Foulis *et al.*, 1994, Example 4.2), is lattice ordered (in fact, it is a chain), and it is the smallest effect algebra that is not an orthoalgebra.

The following theorem, the verification of which is straightforward, makes it clear why the logical structures associated with physical systems are virtually guaranteed to be effect algebras. Indeed, the main reason we elected to begin with general orthostructures (Definition 2.1), rather than directly introducing effect algebras, is to emphasize that, whereas all of the definitions make sense in considerable generality, the actual structures of scientific interest are in fact the nondegenerate regular effect algebras.

Theorem 2.11. If $(L, 0, u, 1, \oplus)$ is an orthostructure with a nonempty, full, and separating set of probability measures, then L is a nondegenerate regular effect algebra.

Henceforth, in view of Theorem 2.11, we shall restrict our attention to effect algebras, eventually arguing that the interesting ones are the interval algebras in a unigroup. Note that both conditions (ii) and (iii) of Definition 2.1 can be deduced from the other assumptions in the definition of an effect algebra.

Let L be an effect algebra and let Λ be a set of probability measures on L. Then Λ is full iff it is *order determining* in the sense that, for all p, q

$\in L$, $p \leq q$ iff $\omega(p) \leq \omega(q)$ for all $\omega \in \Lambda$. Therefore, if Λ is full, it is automatically separating.

3. INTERVAL EFFECT ALGEBRAS

If G is an additively written, partially ordered abelian group (Goodearl, 1986), we denote the *positive cone* in G by $G^+ := \{g \in G | 0 \leq g\}$. The *standard positive cone* in the additive group $\mathbb{R}$ of real numbers, (totally) ordered in the usual way, is denoted by $\mathbb{R}^+$. Likewise, the additive group $\mathbb{Z}$ of integers is (totally) ordered by the standard positive cone $\mathbb{Z}^+ = \mathbb{Z} \cap \mathbb{R}^+$.

Definition 3.1. If G is a partially ordered abelian group and $u \in G^+$, define the *interval* $G^+[0, u] := \{g \in G | 0 \leq g \leq u\}$. Then the orthostructure $L := G^+[0, u]$, as in Example 2.3, is an effect algebra. An effect algebra isomorphic to such an effect algebra is called an *interval effect algebra*.

In Definition 3.1, the orthosupplementation on $G^+[0, u]$ is given by $x' := u - x$ and the effect-algebra partial order is the restriction to $G^+[0, u]$ of the partial order on G.

Theorem 3.2. A sub-effect algebra of an interval effect algebra is again an interval effect algebra.

Proof. Bennett and Foulis (1997, Corollary 2.5). ■

As a consequence of Theorem 2.11 and the following companion theorem, *nearly every structure that has been proposed as the logic of a physical system is an interval effect algebra.*

Theorem 3.3. If an effect algebra L has a nonempty and full set of probability measures, it is a nondegenerate, regular, interval effect algebra.

Proof. By Foulis and Bennett (1994, Theorem 9.6, Part (v)), L is an interval effect algebra, and by Theorem 2.11, it is regular and nondegenerate. ■

Definition 3.4. Let G be a partially ordered abelian group with positive cone G^+ and let $u \in G^+$.

(i) G^+ is said to be a *generating cone* in G iff $G = G^+ - G^+$.

(ii) u is a *generative* element of G iff G^+ is a generating cone and every element in G^+ is a sum of a finite sequence of elements of $G^+[0, u]$.

(iii) u is an *order unit* in G iff, for every $g \in G$, there exists $n \in \mathbb{Z}^+$ such that $g \leq nu$.

(iv) u is *universal* in G iff, for every abelian group K, every K-valued measure ϕ: $G^+[0, u] \to K$ can be extended to a group homomorphism ϕ^*: $G \to K$.

To say that u is a generative element of G means that the interval $G^+[0, u]$ generates G^+ as a semigroup and it generates G as a group. If u is generative, then it is an order unit, and, if G admits an order unit, then G^+ is a generating cone.

Example 3.5. Let $G := \mathbb{Z}$ as an additive abelian group, but with the nonstandard cone $G^+ := \{3m + 5n | m, n \in \mathbb{Z}^+\}$. Then 8 is a generative element of G^+, but it is not universal.

Lemma 3.6. If G is a lattice-ordered abelian group, then any order unit u in G^+ is generative and universal.

Proof. That u is generative follows from the Riesz decomposition property of a lattice-ordered group (Goodearl, 1986). That u is universal is a consequence of Bennett and Foulis (1995, Theorem 4.14). ■

In the so-called operational (or base-norm/order-unit) approach to the foundations of quantum mechanics (Davies and Lewis, 1970; Edwards, 1970), effects are represented by elements of the order interval $V^+[0, u]$ in a partially ordered real Banach space V with an order unit u. The following lemma connects this approach with the theory of unigroups.

Lemma 3.7. Let V be a partially ordered real vector space with positive cone V^+ and let u be an order unit in V^+. Then, regarding V as an additive abelian group, u is generative and universal.

Proof. That u is generative is clear and that it is universal follows from Bennett and Foulis (1997, Corollary 4.6). ■

4. UNIGROUPS

We have now prepared the way for the following fundamental definition.

Definition 4.1. A *unigroup* is a triple (G, G^+, u) consisting of a partially ordered abelian group G with positive cone G^+ and a generative universal element $u \in G^+$ called the *unit.* If (G, G^+, u) is a unigroup, then $G^+[0, u]$, organized into an interval effect algebra, is called the *unit interval* in (G, G^+, u).

If (G, G^+, u) is a unigroup, then $u = 0$ iff $G = G^+ = \{0\}$. A unigroup in which the unit is zero (and therefore which consists only of the single element 0) is said to be *degenerate.*

Definition 4.2. Let (G, G^+, u) and (H, H^+, v) be unigroups. A group homomorphism $\phi: G \to H$ is *positive* iff $\phi(G^+) \subseteq H^+$, and it is *normalized* iff $\phi(u) = v$. A *unigroup homomorphism* $\phi: G \to H$ is a positive normalized group homomorphism.

Unless confusion threatens, we say that G is a unigroup with unit u rather than saying that (G, G^+, u) is a unigroup. Unigroup isomorphisms and unigroup automorphisms are defined in the obvious ways. We omit the straightforward proof of the next lemma.

Lemma 4.3. Let G and H be unigroups with units u and v, respectively. Then there is a one-to-one correspondence $\alpha \leftrightarrow \alpha^*$ between effect-algebra morphisms $\alpha: G^+[0, u] \to H^+[0, v]$ and unigroup homomorphisms $\alpha^*: G \to H$ such that α^* is an extension of α to G. Furthermore, α is an effect-algebra isomorphism iff α^* is a unigroup isomorphism.

Corollary 4.4. If G is a unigroup, then there is a one-to-one correspondence $\alpha \leftrightarrow \alpha^*$ between effect-algebra automorphisms α of $G^+[0, u]$ and unigroup automorphisms α^* of G such that α^* is an extension of α.

For the proof of the next theorem, see Bennett and Foulis (1997, Corollary 4.2).

Theorem 4.5. If L is an interval effect algebra with unit u, there is a unigroup G with unit u such that $L = G^+[0, u]$.

Definition 4.6. The unigroup of Theorem 4.5, which is uniquely determined by L up to an isomorphism by Lemma 4.3, is called the *universal group* (or simply the *unigroup*) for the interval effect algebra L.

By Corollary 4.4, the group Aut(L) of effect-algebra automorphisms of an interval effect algebra L is isomorphic to the group Aut(G) of unigroup automorphisms of its universal group.

Lemma 4.7. If (G, G^+, u) is a unigroup and $0 \neq n \in \mathbb{Z}^+$, then (G, G^+, nu) is again a unigroup.

Proof. See Foulis *et al.* (1994, Lemma 3.5). ■

Definition 4.8. If $L = G^+[0, u]$ is an interval effect algebra with unigroup (G, G^+, u), $0 \neq n \in \mathbb{Z}^+$, then $nL := G^+[0, nu]$. In particular, the interval effect algebra $2L$ is said to be obtained from L by *doubling*.

If $0 \neq n \in \mathbb{Z}^+$, the interval effect algebra $C_n := \mathbb{Z}^+[0, n]$ is called the *n-chain*. If L is an interval effect algebra, it is not difficult to show that nL is isomorphic to the tensor product $C_n \otimes L$ in the category of interval effect algebras (Foulis *et al.*, 1994, Section 9).

5. PRODUCTS AND COPRODUCTS OF UNIGROUPS

With unigroups as objects and unigroup homomorphisms as morphisms, we obtain a category. In Foulis *et al.* (1994) we showed that this category admits finite products, coproducts, and tensor products. We now show that it actually admits arbitrary products and coproducts. (Note, however, that the categorical products and coproducts of unigroups do not in general coincide with the direct product and direct sums of the underlying groups.)

Theorem 5.1. The category of unigroups admits arbitrary products and the categorical product of the universal groups of a family of interval effect algebras is the universal group of the cartesian product of the effect algebras.

Proof. Let (G_i, G_i^+, u_i) be a nonempty family of unigroups, denote by $\times_i G_i$ the cartesian product of the abelian groups G_i, organize $\times_i G_i$ into a partially ordered abelian group with positive cone $(\times_i G_i)^+ := \{g \in \times_i G_i | g_i \in G_i^+ \text{ for all } i\}$, and let $u \in (\times_i G_i)^+$ be defined by $u(i) = u_i$ for all i. Let $L_i := G_i^+[0, u_i]$ for all i, and let $L = \times_i L_i$ be organized into an effect algebra with coordinatewise operations. Evidently, $L = (\times_i G_i)^+[0, u]$. If the family (G_i) is finite, then $(\times_i G_i, (\times_i G_i)^+, u)$ is a unigroup, and it is the universal group for L, but if the family (G_i) is infinite, this need not be the case.

Even in the infinite case, the fact that $L = (\times_i G_i)^+[0, u]$ shows that L is an interval effect algebra, so by Theorem 4.5, there is a unigroup (G, G^+, u) with $L = G^+[0, u]$. For each i, the mapping $L \to L_i$ given by $p \mapsto p(i)$ extends to a unigroup homomorphism π_i: $G \to G_i$. We claim that, with the projection mappings π_i, G is the product of the unigroups G_i. Indeed let (H, H^+, v) be a unigroup with $M = H^+[0, v]$ and let ϕ_i: $H \to G_i$ be unigroup homomorphisms for all i. Define $\phi: M \to \times_i G_i$ by $\phi(q)(i) := \phi_i(q)$ for all i and all $q \in M$. Since $\phi: M \to L \subseteq G$ is a G-valued measure, there is a unigroup homomorphism ϕ^*: $H \to G$ such that $\phi^* = \phi$ on M. Thus, for all $q \in M$, we have $\pi_i(\phi^*(q)) = \pi_i(\phi(q)) = \phi(q)(i) = \phi_i(q)$, so $\pi_i \circ \phi^* = \phi_i$ on M. Since M generates H as a group and both $\pi_i \circ \phi^*$ and ϕ_i are group homomorphisms, it follows that $\pi_i \circ \phi^* = \phi_i$ on H. ■

Consider a nonempty family (L_i) of nondegenerate effect algebras and let L be an effect algebra with unit u. A family (α_i) of injective effect-algebra morphisms α_i: $L_i \to L$ is said to provide a representation of L as a *horizontal sum* of the family (L_i) iff the following conditions hold: (i) For each i, $\alpha_i(L_i)$ is a sub-effect algebra of L. (ii) For each i, α_i: $L_i \to \alpha_i(L_i)$ is an effect-algebra isomorphism. (iii) $L = \cup_i \alpha_i(L_i)$. (iv) $i \neq j \Rightarrow \alpha_i(L_i) \cap \alpha_i(L_j) = \{0, u\}$. (v) $i \neq j$, $p \in \alpha_i(L_i)$, $q \in \alpha_j(L_j)$, and $p \perp q \Rightarrow p = 0$ or $q = 0$.

If L is represented as a horizontal sum of the family (L_i) by the injective effect-algebra morphisms α_i: $L_i \to L$, it is clear that L is a coproduct of the family (L_i) in the category of effect algebras with respect to the morphisms α_i.

Any nonempty family (L_i) of nondegenerate effect algebras has a horizontal sum L with unit u, constructed as follows: For each i, form an isomorphic copy E_i of L_i in such a way that, for the family (E_i), all the zeros coincide, all the units coincide and are equal to u, and $i \neq j \Rightarrow E_i \cap E_j = \{0, u\}$. Let $L := \cup_i E_i$. For $p, q \in L$, say that $p \perp q$ iff there is an index i with $p, q \in E_i$ and $p \perp q$ in E_i, in which case $p \oplus q$ is defined to be $p \oplus q$ as calculated in E_i. Then the effect-algebra isomorphisms α_i: $L_i \to E_i$ provide injective effect-algebra morphisms α_i: $L_i \to L$ with respect to which L is represented as a horizontal sum of the family (L_i).

Theorem 5.2. Let (G_i, G_i^+, u_i) be a nonempty family of nondegenerate unigroups. Then there exists a unigroup (G, G^+, u) and unigroup homomorphisms $\alpha_i^* : G_i \to G$ such that the restrictions α_i of α_i^* to $L_i := G_i^* [0, u_i]$ provide a representation of $L := G^+ [0, u]$ as a horizontal sum of the family (L_i) and, with respect to the unigroup homomorphisms (α_i^*), G is a coproduct of the family (G_i) in the category of unigroups.

Proof. The proof proceeds along the same lines as the proof of Foulis *et al.* (1994, Theorem 8.1), so we merely sketch it here. Let $\Sigma_i G_i$ be the direct sum of the groups G_i. By relabeling the groups G_i if necessary, we can assume that each G_i is a subgroup of $\Sigma_i G_i$ and that every element $g \in \Sigma_i G_i$ can be written uniquely in the form $g = \Sigma_i g_i$, where $g_i \in G_i$ for all i and $g_i = 0$ for all but possibly finitely many indices i. Partially order $\Sigma_i G_i$ in such a way that $g = \Sigma_i g_i \in (\Sigma_i G_i)^+$ iff every $g_i \in G_i^+$.

Let K be the subgroup of $\Sigma_i G_i$ consisting of all elements of the form $k = \Sigma_i n_i u_i$ such that $n_i \in \mathbb{Z}$, $n_i = 0$ for all but possibly finitely many indices i, and $\Sigma_i n_i = 0$. Evidently $K \cap (\Sigma_i G_i)^+ = \{0\}$. Let $G := (\Sigma_i G_i)/K$ and let $\eta : \Sigma_i G_i \to G$ be the canonical group epimorphism. Since $K \cap (\Sigma_i G_i) = \{0\}$, G is a partially ordered abelian group with positive cone $G^+ := \eta((\Sigma_i G_i)^+)$. Define $u \in G$ by $u := \eta(u_i)$, noting that u is independent of the choice of the index i and $0 \neq u \in G^+$. Let $L := G^+ [0, u]$.

For each index i, let α_i^* be the restriction of η to the subgroup G_i of $\Sigma_i G_i$, noting that $\alpha_i^*: G_i \to G$ is a positive normalized group homomorphism. Furthermore, let α_i be the restriction of α_i^* to $L_i = G_i^+ [0, u_i]$, noting that $\alpha_i : L_i \to G^+ [0, u]$ is an effect-algebra morphism. In fact, $\alpha_i(L_i)$ is a sub-effect algebra of L and α_i is an effect-algebra isomorphism of L_i onto $\alpha_i(L_i)$ for each index i. Furthermore, (G, G^+, u) is a unigroup and it is a coproduct with respect to the unigroup homomorphisms (α_i^*) in the category of unigroups. Finally, the injective effect-algebra morphisms (α_i) provide a representation of L as a horizontal sum of the family (L_i). ■

6. HEYTING EFFECT ALGEBRAS AND HEYTING UNIGROUPS

In ascribing order-theoretic properties to an effect algebra L, we always have reference to the partially ordered set $(L, \leq)$ where $\leq$ is defined as in Part (ii) of Definition 2.9. The greatest lower bound (or meet) and least upper bound (or join) of elements $p, q \in L$, when they exist, are written as $p \wedge q$ and $p \vee q$, respectively. To say that L is *lattice ordered* means that $(L, \leq)$ is a lattice. If $(L, \leq)$ is a distributive lattice, we say that L is a *distributive* effect algebra.

Not every lattice-ordered effect algebra is an interval effect algebra. For instance, there are orthomodular lattices that admit no probability measures (Greechie, 1971), whereas every interval effect algebra admits at least one probability measure (Bennett and Foulis, 1997, Theorem 5.5). Just because an interval effect algebra is lattice ordered, its universal group need not be lattice ordered, nor even an interpolation group (Ravindran, 1996). Although every finite distributive effect algebra is an interval effect algebra (Greechie *et al.*, 1995, Corollary 7.10), we do not know whether every distributive effect algebra is an interval effect algebra.

If L is an effect algebra, then the *center* $C(L)$ of L (Greechie *et al.*, 1995) consists of those elements $z \in L$ such that, for every $x \in L$, the elements $x_1 := x \wedge z$ and $x_2 := x \wedge z'$ exist and the mapping $x \mapsto (x_1, x_2)$ decomposes L into a cartesian product of effect algebras L_1 and L_2. The center $C(L)$ is a sub-effect algebra of L and, as an effect algebra in its own right, it is a boolean algebra.

Recall that a meet semilattice L with 0 is said to be *pseudocomplemented* iff there is a mapping $p \mapsto p^{\sim}$ on L such that, for all $p, q \in L$, $p \wedge q = 0 \Leftrightarrow q \leq p^{\sim}$ (Birkhoff, 1967). If L is pseudocomplemented and $p, q \in L$, then $p \leq q \Rightarrow q^{\sim} \leq p^{\sim}$, $p \leq p^{\sim\sim}$, $p^{\sim\sim\sim} = p^{\sim}$, $p \wedge q = 0 \Leftrightarrow p \wedge q^{\sim\sim} = 0$, $(p \wedge q)^{\sim\sim} = p^{\sim\sim} \wedge q^{\sim\sim}$, L has a largest element $0^{\sim}$, and $0^{\sim\sim} = 0$. Furthermore, L satisfies the De Morgan law for arbitrary joins, that is, if (p_i) is a family of elements of L and $\vee_i p_i$ exists in L, then $\wedge_i p_i^{\sim}$ exists in L and $(\vee_i p_i)^{\sim} = \wedge_i p_i^{\sim}$. An element p of a pseudocomplemented meet semilattice L is *closed* iff $p = p^{\sim\sim}$ and *dense* iff $p^{\sim} = 0$. If (p_i) is a family of closed elements of L and $\wedge_i p_i$ exists in L, then $\wedge_i p_i$ is closed.

Definition 6.1. A *Heyting effect algebra* (HEA) is a lattice-ordered effect algebra L with a center-valued pseudocomplementation $p \mapsto p^{\sim} \in C(L)$. A unigroup (G, G^+, u) is a *Heyting unigroup* iff $G^+[0, u]$ is an HEA.

Parts (i) and (ii) of the following lemma are two of the conditions in the definition of a BZ-poset (Cattaneo and Nistico, 1989).

Lemma 6.2. If L is an HEA and $p \in L$, then:

(i) $p^{\sim} \leq p'$.

(ii) $p^{\sim\prime} = p^{\sim\sim}$.

(iii) $p \in C(L)$ iff $p^{\sim} = p'$.
(iv) $C(L) = \{p^{\sim} | p \in L\}$.

Proof. (i) Since $p^{\sim} \in C(L)$, $p = (p \wedge p^{\sim}) \oplus (p \wedge p^{\sim\prime}) = p \wedge p^{\sim\prime}$, so $p \leq p^{\sim\prime}$, whence $p^{\sim} \leq p'$. (ii) Since $p^{\sim} \in C(L)$, it follows that $p^{\sim\prime} \wedge p^{\sim} = 0$, so $p^{\sim\prime} \leq p^{\sim\sim}$. By (i), $(p^{\sim})^{\sim} \leq (p^{\sim})'$, whence $p^{\sim\prime} = p^{\sim\sim}$. (iii) If $p' = p^{\sim}$, then $p' \in C(L)$, so $p = p'' \in C(L)$. Conversely, suppose $p \in C(L)$. Then, for all $q \in L$, $q = (q \wedge p) \oplus (q \wedge p')$, so $q \wedge p = 0$ iff $q \leq p'$, whence $p^{\sim} = p'$. (iv) We have only to prove that $C(L) \subseteq \{p^{\sim} | p \in L\}$. Let $q \in C(L)$. Then $q' \in C(L)$, so $q = (q')' = q'^{\sim}$ by (iii), whence $q \in \{p^{\sim} | p \in L\}$. ■

A phi-symmetric effect algebra (Bennett and Foulis, 1995) can be defined as a lattice-ordered effect algebra with the Riesz decomposition property.

Theorem 6.3. If L is an HEA, then L is a phi-symmetric effect algebra.

Proof. If $p, q \in L$ with $p \wedge q = 0$, then $q \leq p^{\sim} \leq p'$ by Part (i) of Lemma 6.2. Therefore L is phi-symmetric by (Bennett and Foulis (1995, Theorem 3.11, Part (vii)). ■

Corollary 6.4. If L is an HEA, then L is a distributive interval effect algebra and its unigroup is lattice ordered.

Proof. By Theorem 6.3 and Bennett and Foulis (1995, Theorem 3.14), L is distributive. By Ravindran (1996, Theorem 3.9), L is an interval effect algebra with a lattice-ordered universal group. ■

Theorem 6.5. If L is an HEA, then L is a BZ-lattice.

Proof. In view of Lemma 6.2, we have only to prove that L is regular (Cattaneo and Nistico, 1989). By Corollary 6.4, there is a lattice-ordered group G such that $L = G^{+}[0, u]$. By Goodearl (1986, Proposition 1.22), if $g \in G$ and $2g \in G^{+}$, then $g \in G^{+}$. Suppose p and q are isotropic elements of L. Then $2p, 2q \leq u$, so $0 \leq (u - 2p) + (u - 2q) = 2(u - (p + q))$, whence $p + q \leq u$, that is, $p \perp q$. ■

Definition 6.6. Let L be an effect algebra with center $C(L)$ and let $p \in L$. If there is a smallest element c in the set $\{z \in C(L) | p \leq z\}$, then c is called the *central cover* of p in L, denoted by $^{\gamma}p := c$. Say that L has the *central cover property* iff every element in L has a central cover.

Lemma 6.7. If L is an HEA, then L has the central cover property and for all $p, q \in L$:
(i) $\gamma p = p^{\sim\sim} = p^{\sim\prime}$.
(ii) $\gamma(p \wedge q) = \gamma p \wedge \gamma q$.
(iii) $\gamma p \perp q \Leftrightarrow p \wedge q = 0 \Leftrightarrow p \perp \gamma q$.

(iv) $\gamma p - p = \gamma p \wedge p'$.

(v) If (p_i) is a family of elements of L and $\vee_i p_i$ exists in L, then $\vee_i \gamma p_i$ exists in L and $\gamma(\vee_i p_i) = \vee_i \gamma p_i$.

(vi) The center $C(L)$ is closed under the formation of all existing meets and joins in L.

Proof. (i) We have $p \leq p^{\sim\sim} = p^{\sim\prime} \in C(L)$. If $p \leq z \in C(L)$, then $z = z^{\sim\sim}$ by Part (iv) of Lemma 6.2, so $p^{\sim\sim} \leq z^{\sim\sim} = z$.

(ii) Since $(p \wedge q)^{\sim\sim} = p^{\sim\sim} \wedge q^{\sim\sim}$, (i) implies that ${}^{\gamma}(p \wedge q) = {}^{\gamma}p \wedge {}^{\gamma}q$.

(iii) $\gamma p \perp q \Leftrightarrow p^{\sim\sim} \leq q' \Leftrightarrow q \leq p^{\sim\sim\prime} \Leftrightarrow q \leq p^{\sim\sim\sim} \Leftrightarrow q \leq p^{\sim} \Leftrightarrow p \wedge q = 0$. By symmetry, $p \perp \gamma q \Leftrightarrow p \wedge q = 0$.

(iv) For elements $x \perp y$ in any effect algebra, we have $(x \oplus y)' = x' - y$. Therefore, with $x = p^{\sim\sim} \wedge p'$ and $y = p$,

$$((p^{\sim\sim} \wedge p') \oplus p)' = (p^{\sim\sim\prime} \vee p'') - p = (p^{\sim\sim\sim} \vee p) - p$$
$$= (p^{\sim} \vee p) - p$$

Because $p^{\sim} \leq p'$ and $p^{\sim} \wedge p = 0$, Bennett and Foulis (1995, Lemma 3.1) implies that $p^{\sim} \vee p = p \oplus p$, and it follows that

$$((p^{\sim\sim} \wedge p') \oplus p)' = (p^{\sim} \oplus p) - p = p^{\sim}$$

whence $(p^{\sim\sim} \wedge p') \oplus p = p^{\sim\prime} = p^{\sim\sim}$, so $p^{\sim\sim} - p = p^{\sim\sim} \wedge p'$.

(v) Let $p = \vee_i p_i$. Because $p_i \leq p$ for all i, it follows that $\gamma p_i \leq \gamma p$ for all i. Suppose $q \in L$ and $\gamma p_i \leq q$ for all i. Then $\gamma p_i \perp q'$, so $p_i \perp \gamma(q')$ by (iii), whence $p_i \leq (\gamma(q'))'$ for all i. Thus, $p \leq (\gamma(q'))'$, so $\gamma(q') \perp p$, whence $q' \perp \gamma p$, that is, $\gamma p \leq q$.

(vi) Let (p_i) be a family of elements in $C(L)$, so that $p_i = \gamma p_i$ for all i. If $p = \vee_i p_i$, then ${}^{\gamma}p = \vee_i \gamma p_i = \vee_i p_i = p$ by (v). Suppose $q = \wedge_i p_i$. Then $q \leq p_i$ for all i, whence $\gamma q \leq \gamma p_i = p_i$ for all i, so $\gamma q \leq q$. But $q \leq \gamma q$, so $q = \gamma q \in C(L)$. ■

A *Stone algebra* is a distributive lattice with smallest and largest elements 0 and 1 and with a pseudocomplementation $p \mapsto p^{\sim}$ such that $p^{\sim} \vee p^{\sim\sim} = 1$ for all $p \in L$. In a Stone algebra L, one has both of the De Morgan laws $(p \vee q)^{\sim} = p^{\sim} \wedge q^{\sim}$ and $(p \wedge q)^{\sim} = p^{\sim} \vee q^{\sim}$ (Gratzer, 1978).

Theorem 6.8. If L is an HEA, then L is a Stone algebra.

Proof. Let u be the unit of L. Because $p^{\sim\sim} = p^{\sim\prime}$ is the complement of $p^{\sim}$ in $C(L)$, we have $u = p^{\sim} \vee p^{\sim\sim}$. ■

Theorem 6.9. Let L be a lattice-ordered effect algebra. Then L is an HEA iff L admits a central cover mapping $p \mapsto \gamma p$ such that, for all $p, q \in L$, $p \wedge q = 0 \Rightarrow \gamma p \wedge q = 0$.

Proof. If L is an HEA, and $p, q \in L$ with $p \wedge q = 0$, then $q \leq p^{\sim} = p^{\sim\sim\sim} = (\gamma p)^{\sim}$, whence $\gamma p \wedge q = 0$. Conversely, suppose L is lattice ordered and admits a central cover mapping with the given property. Define $p^{\sim} := (\gamma p)'$ for all $p \in L$. Since $\gamma p \in C(L)$, it follows that $p^{\sim} \in C(L)$ and $p^{\sim}$ is a complement of γp in L. Hence, $p \wedge p^{\sim} \leq \gamma(p) \wedge p^{\sim} = 0$, Suppose $q \in L$ and $p \wedge q = 0$. Then, by hypothesis, $\gamma p \wedge q = 0$. Since $\gamma p \in C(L)$, we have $q = (q \wedge \gamma p) \oplus (q \wedge (\gamma p)') = q \wedge p^{\sim}$, whence $q \leq p^{\sim}$. ■

In the proof of the following theorem, we use the fact that if L is any effect algebra, $x, y \in L$ with $x \leq y$, and $z \in C(L)$, then $(y - x) \wedge z$, $y \wedge z$, and $x \wedge z$ exist in L and $(y - x) \wedge z = (y \wedge z) - (x \wedge z)$ (Greechie *et al.*, 1995).

Theorem 6.10 (Generalized Comparability). If L is an HEA, $p, q \in L$, and $z = (p - p \wedge q)^{\sim}$, then $p \wedge z \leq q$ and $q \wedge z' \leq p$.

Proof. We have $z \in C(L)$ and $z' = z^{\sim} \in C(L)$. Thus,

$$(p \wedge z) - (p \wedge z) \wedge (q \wedge z) = (p - p \wedge q) \wedge z = 0$$

whence $p \wedge z = (p \wedge z) \wedge (q \wedge z)$, so $p \wedge z \leq q \wedge z$. Also, by Bennett and Foulis (1995, Corollary 2.4),

$$(q - p \wedge q) \wedge (p - p \wedge q) = 0, \qquad \text{so} \quad q - p \wedge q \leq z$$

and it follows that

$$(q \wedge z') - (p \wedge z') \wedge (q \wedge z') = (q - p \wedge q) \wedge z' \leq z \wedge z' = z \wedge z^{\sim} = 0$$

so $q \wedge z' = (p \wedge z') \wedge (q \wedge z')$, whence $q \wedge z' \leq p \wedge z'$. ■

The following technical lemma will assist in our proof of Theorem 6.13 below.

Lemma 6.11. If L is an HEA, $p, q, r \in L$, $r \wedge p \leq q \leq p$, and $r \leq (p - q)^{\sim\sim}$, then $r \leq q$.

Proof. Assume the hypotheses and let $z := (p - p \wedge r)^{\sim}$. Then $p \wedge z \leq r$ by Theorem 6.10, whence

$$p \wedge z = r \wedge p \wedge z \leq q \wedge z \leq p \wedge z$$

so $p \wedge z = q \wedge z$. Since $z \in C(L)$, we have

$$(p - q) \wedge z = (p \wedge z) - (q \wedge z) = 0$$

whence $p - q \leq z^{\sim}$, from which it follows that

$$r \leq (p - q)^{\sim\sim} \leq z^{\sim\sim\sim} = z^{\sim} = z'$$

Consequently, $r = r \wedge z' \leq p$ by Theorem 6.10, and it follows that $r = r \wedge p \leq q$. ■

Definition 6.12. If L is an HEA and $p, q \in L$, define $(p \supset q) \in L$ by $(p \supset q) := (p - p \wedge q)^{\sim} \vee q$.

Theorem 6.13. If L is an HEA, then L is a Heyting algebra with $(p, q) \mapsto p \supset q$ as the Heyting implication connective; that is, for $p, q, r \in L$, $r \wedge p \leq q \Leftrightarrow r \leq (p \supset q)$.

Proof. Let $z := (p - p \wedge q)^{\sim}$, so that $(p \supset q) = z \vee q$, $z \in C(L)$, and $z' = z^{\sim} \in C(L)$. For each element $x \in L$, let $x_1 := x \wedge z$ and $x_2 = x \wedge z'$. Let $L_i := \{x_i | x \in L\}$ for $i = 1, 2$. Since $z \in C(L)$ is isomorphic to $L_1 \times L_2$ under $x \mapsto (x_1, x_2)$.

By Theorem 6.10, $p_1 = p \wedge z \leq q \wedge z = q_1$ and $q_2 = q \wedge z' \leq p \wedge z' = p_2$. Also, $z_1 = z$ and $z_2 = z \wedge z' = 0$.

Suppose $r \leq (p \supset q) = z \vee q$. To prove $r \wedge p \leq q$, it suffices to prove $r_i \wedge p_i \leq q_i$ for $i = 1,2$. Since $p_1 \leq q_1$, we have $r_1 \wedge p_1 \leq q_1$. Also, $r \leq z \vee q$ implies $r_2 \leq z_2 \vee q_2 = q_2$, so $r_2 \wedge p_2 \leq q_2$, and we have $r \wedge p \leq q$.

Conversely, suppose $r \wedge p \leq q$. To prove that $r \leq z \vee q$, it suffices to prove $r_i \leq z_i \vee q_i$ for $i = 1, 2$. Since $r_1 \leq z = z_1 \leq z_1 \vee q_1$, we need only prove that $r_2 \leq z_2 \vee q_2$; that is, $r_2 \leq q_2$. Since $q_2 \leq p_2$, we have $q_2 = p_2 \wedge q_2 = p \wedge q \wedge z'$. Consequently,

$$\begin{aligned} p_2 - q_2 &= (p \wedge z') - (p \wedge q \wedge z') = (p - p \wedge q) \wedge z' \\ &= (p - p \wedge q) \wedge z^{\sim} = (p - p \wedge q) \wedge (p - p \wedge q)^{\sim\sim} \\ &= p - p \wedge q \end{aligned}$$

Therefore, $z = (p - p \wedge q)^{\sim} = (p_2 - q_2)^{\sim}$. The condition $r \wedge p \leq q$ implies that $r_2 \wedge p_2 \leq q_2$, whence $r_2 \wedge p_2 \leq q_2 \leq p_2$. Also, $r_2 \leq z' = (p_2 - q_2)^{\sim\prime} = (p_2 - q_2)^{\sim\sim}$, and it follows from Lemma 6.11 that $r_2 \leq q_2$. ■

7. STONE GROUPS AND CONDITIONALIZATION

If X is a topological space, A is a subgroup of the additive group $\mathbb{R}$ of real numbers, and A is endowed with the relative topology inherited from $\mathbb{R}$, then $C(X, A)$ denotes the additive group, under pointwise addition, of all continuous functions f: $X \rightarrow A$ with the pointwise partial order. Evidently, $C(X, A)$ is lattice ordered, every order unit u in $C(X, A)$ is a strictly positive function, and if X is compact, every strictly positive function u in $C(X, A)$ is an order unit. If u is an order unit in $C(X, A)$, then $(C(X, A), u)$ is a unigroup.

Definition 7.1. If X is a topological space and A is an additive subgroup of $\mathbb{R}$, let $F(X, A)$ be the subgroup of $C(X, A)$ consisting of the functions $f \in F(X, A)$ such that $f(X) := \{f(x) | x \in X\}$ is a finite subset of A.

In Definition 7.1, it is clear that, as a partially ordered set, $F(X, A)$ is a sublattice of $C(X, A)$, so $F(X, A)$ is a lattice-ordered abelian group.

Theorem 7.2. If X is a topological space, A is an additive subgroup of $\mathbb{R}$, and $u \in F(X, A)$ with $0 < u(x)$ for all $x \in X$, then $(F(X, A), u)$ is a Heyting unigroup.

Proof. Since the functions in $F(X, A)$ take on only finitely many values, it is clear that u is an order unit. An order unit in a lattice-ordered abelian group is automatically generative and universal, so $(F(X, A),u)$ is a unigroup. Let $L := F(X, A)^+[0, u]$. Since L is a lattice-ordered effect algebra, Corollary 3.17 of Bennett and Foulis (1995) implies that $C(L)$ consists of all functions $f \in L$ such that, for all $x \in X$,

$$(f \wedge (u - f))(x) = \min(f(x), u(x) - f(x)) = 0$$

Hence, $f \in C(L)$ iff, for all $x \in X$, $f(x) = 0$ or $f(x) = u(x)$. For $f \in L$, let $f^{\sim}(x) := 0$ iff $f(x) \neq 0$ and $f^{\sim}(x) := u(x)$ iff $f(x) = 0$. Thus, $f \in L \Rightarrow f^{\sim} \in C(L)$ and, for all $g \in L$, $f \wedge g = 0$ iff $g \leq f^{\sim}$, whence L is an HEA. ∎

Definition 7.3. If X is a compact, Hausdorff, totally disconnected topological space (i.e., a Stone space), then the partially ordered abelian group $G(X) := C(X, \mathbb{Z})$ is called the *Stone group* over X.

For the remainder of this section, we assume that *X is a Stone space and $G(X)$ is the corresponding Stone group.* Let B be the boolean algebra of compact open subsets of X. By the Stone representation theorem (Stone, 1936, 1937), any boolean algebra B can be so represented.

Since $\mathbb{Z}$ carries the discrete topology, every function $g \in G(X)$ satisfies the condition that $g^{-1}(n)$ is a clopen subset of X for every $n \in \mathbb{Z}$; hence, since X is compact, g can take on only finitely many different values. Thus, $G(X) = F(X, \mathbb{Z})$. Denote by $u_1 \in G(X)^+$ the constant function $u_1(x) := 1$ for all $x \in X$, noting that u_1 is an order unit and that $u \in G(x)^+$ is an order unit iff $u_1 \leq u$. Therefore a Stone group $G(X)$ admits a unique smallest order unit, namely u_1.

The effect algebra $L_1 := G(X)^+ [0, u_1]$ consists of the characteristic set functions of compact open subsets of X, hence it is isomorphic to the boolean algebra B. In what follows, we *shall identify B with L_1*. If u is an order unit in $G(X)^+$, then, as observed in the proof of Theorem 7.2, elements in the center of $G(X)^+ [0, u]$ are precisely the functions that agree with u wherever they are nonzero; hence, the center of $G(X)^+ [0,u]$ is isomorphic to $L_1 = B$. Thus, the centers of all the Heyting effect algebras corresponding to order units $u \in G(X)^+$ are isomorphic to each other and to the boolean algebra B. The dense elements in the Heyting algebra $G(X)^+ [0, u]$ are precisely the functions that are strictly positive.

If n is a positive integer, let $L_n := G(X)^+[0, nu_1]$, noting that $L_n = nL_1 = nB$ in the notation of Definition 4.8. If one regards B as a "logic of two-valued propositions" or a "space of events," then the elements of nB can be regarded as an "$(n + 1)$-valued logic" over B. Indeed, each $x \in X$ produces "truth values" on a discrete scale C_n from 0 to n for the elements $p \in L_n$ by the evaluation $p \mapsto p(x)$.

The effect algebra $2B = L_2$ obtained by doubling the boolean algebra B is a Heyting algebra in which the dense elements g are the functions taking on only the two values 1 or 2 in $\mathbb{Z}$; hence, as a partially ordered set, they form a boolean algebra isomorphic to B under the mapping $g \mapsto g - u_1$. By a theorem of Walker (1994, Theorem 4), it follows that $2B$ can be identified as the space of *conditional events* over the boolean algebra B. Specifically, the identification is accomplished as follows:

Definition 7.4. If $B = L_1$, $2B = L_2$, and $p, q \in L_1$, the *conditionalized element* $p|q \in 2B$ is defined by

$$p|q := (p \wedge q) + (p \vee (u_1 - q))$$

If elements p, q of the boolean algebra B are regarded in the usual way as (classical) propositions or events, then $p|q$ in Definition 7.4 can be thought of as the conditional proposition or event "p given q" (Goodman, 1994). The mapping $p \mapsto p|u_1 = 2p$ provides an isomorphism of B onto the center $C(L_2)$, permitting an identification of the unconditional event p with p conditioned by the sure event u_1. Various logical desiderata such as "modus ponens,"

$$(q|u_1) \wedge (p|q) \le p|u_1$$

and "entailment within the consequent,"

$$(p|q) \wedge (r|q) \le (p \wedge r)|q$$

are now easily derived. Furthermore, $2B$ becomes a 3-valued logic with truth values 0,1,2 under evaluations corresponding to the points in the Stone space X.

The algebra of conditional events is critical for dealing with "if-then" rules in expert systems (Nguyen and Walker, 1994). It comes as a pleasant surprise to see that there is a connection between this algebra and quantum logic.

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